Bounding the productivity default shock : Evidence from the The European Sovereign Debt Crisis

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The Productivity Cost of Sovereign Default: Evidence from The European Debt Crisis*

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Abstract

We calibrate the cost of sovereign defaults using a continuous time model, where government default decisions may trigger a change in regime of a TFP stochastic process. We calibrate the model to a sample of European countries from 2009 to 2012. By comparing the estimated drift in default relative to that in no-default, we find that TFP drops by 3.70%. This is broadly consistent with the 5% drop that is typically used in the literature. The model is also consistent with observed drops in GDP and observed growth during recovery, predicts reasonable recovery times and illustrates why fiscal multipliers are small in sovereign debt crises. We use these features to argue for the reliability of our calibrated TFP drop.

Keywords: Default, Sovereign Debt, Financial Markets, Productivity.
JEL codes: E30, E44, G15.

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1 Introduction

Sovereign defaults are relatively common around the world: they disrupt the ability of a country to produce value and may be very costly for the economies that experience them.[1] Those costs can be interpreted as if they were shocks to productivity originating from a default decision. Such costs have been incorporated into the relevant literature as drops in total factor productivity (TFP) consistent with certain key facts, in particular the fall in GDP that countries experience during a default. We therefore label them 'TFP default costs".

After the model of the crisis in Mexico drawn up by Cole and Kehoe (1996) other papers such as Arellano et al. (2012), Cole and Kehoe (2000),Da-Rocha et al. (2013), Conesa and Kehoe (2014) and Conesa and Kehoe (2015), coincide in setting the costs of a default to a fall in TFP of around 5%, but there is little guidance as to whether this number is too big or not, or as to how far TFP could possibly fall in a default episode.

We estimate TFP default shocks from financial data on stock prices and interest rate spreads. Financial markets are forward-looking and reflect new information as it arrives. That information is immediately reflected in prices. Following Cole et al. (2005), we assume that there is a stronger correlation between the stock market and future TFP during a default crisis.[2] Thus, the spot price of an asset reflects the best knowledge about the future prospects of the impact of a default on TFP and the interest rate spread reflects the risk of defaulting on debt and so on.

The European Sovereign Debt Crisis provides a great opportunity to estimate productivity default shocks from financial data, as some European countries experienced large rises in the

[1]As Mendoza and Yue (2012) report, in almost every episode GDP fell below trend, external financing shut down, interest rates peaked, external debt built up and labour input fell dramatically, imposing large potential costs on each economy that experienced default.

[2]They find that the correlation between the U.S. stock market and future productivity (TFP) during the Great Depression of 1929 is 0.75.

risk of default or interest rate spreads on sovereign debt, while their stock prices were falling. We select a sample of European countries from 2009 to 2012, specifically Austria, Belgium, Finland, France, Germany, Ireland, Italy, Netherlands, Portugal and Spain. A key finding in the data is that interest rate spreads on sovereign debt are negatively correlated with the trend in stock prices. These negative correlations mean that financial markets discount the probability of default on the spot.

To estimate productivity default shocks we build a continuous time model of government default decisions. Governments maximise public expenditure, whch is obtained from expected tax revenues where the only source of uncertainty is TFP, which follows a regime-switching Brownian motion. At any given time the only decision that the government has to make is whether to default or not. If the government defaults it triggers a change in the mean and variance of the TFP process. A large negative shock to TFP reduces the probability of repayment, increases the probability of a permanent change in the productivity regime and reduces the value of firms. As a result the model produces a negative correlation between stock prices and risk premiums.

We find that the permanent drop in productivity implicitly discounted by stock prices during the European Sovereign Debt Crisis was 3.70%. Previous literature on private business default (see Davydenko et al. (2012) and Glober (2013)), finds equivalent TFPs in the range of 15-35%. Therefore, our results on cross country productivity costs of default are consistent with the figure used in the main body of literature, i.e. 5%.

Moreover, we show that the productivity drop estimated by our model is consistent with several facts. First, our model predicts GDP falls over the course of sovereign debt crises compatible with rates of growth in the countries in our sample after those crises. Second, the estimated change in productivity regimes produces a distribution of recovery times that is compatible with past episodes, e.g. the 2002 episode in Argentina. Finally, we use the model prediction that more indebted countries have larger default regions to study the relationship between fiscal policies, risk premiums and GDP growth. We

find that a debt expansion of 25% translates into a decreases of .61% in GDP. Therefore, we conclude that attempts to mitigate the effect of the sovereign debt crisis through fiscal expansion have little effect on the economy, i.e. our model predicts small fiscal multipliers during sovereign debt crises. This is yet another observation that seems consistent with the data.

Our paper is related to the literature that uses continuous time and Brownian motion processes to study debt crises. These tools are standard in finance literature, and are becoming increasingly popular in macro debt crisis literature; see for instance Aguiar et al. (2013); Nuño Barrau and Thomas (2015); Na et al. (2015); M. and Vardoulakis (2013); Du and Schreger (2013).

Section 2 below presents the model that we use to estimate the regime-switching parameters of the underlying TFP process. Section 3 presents the data that needed to estimate TFP parameters and feed the model. Section 4 presents the main results and Section 5 concludes.

2 The Model

This paper presents a model of government default decisions. The objective of any government is to maximise government expenditure, assuming that its budget (including interest payments on sovereign bonds) balances at all times. Tax revenues depend on the implementing of a TFP regime-switching stochastic process. If proceeds are low enough it may be in the best interest of the government to default on sovereign bonds, closing bond markets forever and triggering a change in regime of the TFP process. Such a change in regime captures the potential productivity losses in case of a default. Given the structure of the model, we can estimate the parameters of the TFP process using information on stock prices. The stochastic process, the government default decision problem and the implied value of the firm which will be incorporated into the data are specified below.

Productivity Process: the specification of the continuous time regime-switching stochastic process for productivity is key to our model. This process is written as a geometric Brownian motion:

$$dA = \mu_s A dt + \sigma_s A dz_t.$$

As with any Brownian motion, productivity is characterised by a deterministic component and a stochastic component, which is a Wiener process. The drift μ_s is of the deterministic component and the variance σ_s of the Wiener process are functions of the government's default decision $s \in \{d, nd\}$, where $s = nd$ stands for the state of the economy with no default and vice-versa. In case of a default, the productivity drift and the variance switch to a different regime and stay there forever.

Government Problem: the government is the only decision-maker. At all times it faces the following budget constraint equation:

$$\tau A - g + [q(A) - 1]b = 0.$$

where A is the productivity, τ is a tax rate, g is government expenditure, b is the stock of debt, and $q(A)$ is its corresponding price. The immediate objective of the government is to maximise g

$$g = \tau A + [q(A) - 1]b,$$

and at each moment the only decision that the government has to make is whether or not to default.[3] There is no other decision to be made as government expenditure is in fact a stochastic process, where the government's default decision is about choosing what stochastic

[3]This would be similar to assuming a benevolent government that tries to maximise the utility of a representative household with a separable utility function in private and public consumption.

process is to drive expenditures. In case of a default the drift and variance of government expenditure would be $(\mu, \sigma) = (\mu_d, \sigma_d),)$, whereas if the government does not default it would be $(\mu, \sigma) = (\mu_{nd}, \sigma_{nd}).)$. Upon default there are no more decisions to be made, as bond markets are closed forever.

Therefore the government's problem can be written as a Bellman equation:

$$\begin{aligned} W(A) &= \max_{d \in \{0,1\}} \left\{ \tau A + (q(A) - 1)\, b + (1 + r dt)^{-1} E W(A + dA), W^d(A) \right\} \\ & \quad s.t. \;\; \frac{dA}{A} = \mu_{nd} dt + \sigma_{nd} dz, \end{aligned}$$

where $W(A)$ is the value of repaying, made up of the immediate government expenditure and the expected continuation value of repaying.[4] $W^d(A)$ is the value of defaulting, which equals the expected discounted value of government expenditure from the time of default, and is driven by a stochastic process of drift μ_d and variance σ_d:

$$W^d(A) = \int_0^\infty \tau A e^{-(r - \mu_d + \sigma_d^2/2)t} dt = \frac{\tau A}{r - \mu_d + \sigma_d^2/2}.$$

This gives a stationary stopping rule which is a threshold value A_d for productivity. If A falls below this threshold, the government will choose to default and stay in the default region forever.

The government's value of repaying $W(A)$ can be expressed as an ordinary second order differential equation

$$rW(A) = \tau A + [q(A) - 1]b + \mu_{nd} A W'(A) + \frac{\sigma_{nd}^2}{2} A^2 W''(A) \tag{1}$$

[4] We assume neutral agents. For the consequences of assuming agents concerned with the worst case scenario see Araujo (2015).

with a boundary and smooth pasting conditions

$$W(A_d) = \frac{\tau A_d}{r - \mu_d + \sigma_d^2/2},$$

$$W'(A_d) = \frac{\tau}{r - \mu_d + \sigma_d^2/2}.$$

Note that default regions depends on the price of the bond, $q(.)$, which is endogenous in the government's default decision.

Risk Premium: the risk premium is the difference in returns between a bond and a risk free asset

$$\frac{1}{q(A)} - (1 + r)$$

where r is the risk-free rate of return and $q(A)$the price of the bonds issued, which is related to the government's decision through the productivity stochastic process.[5] Using Ito's lemma $q(A)$ can be found as a solution to a partial differential equation:

$$rq(A) = \mu_{nd} A q'(A) + \frac{\sigma_{nd}^2}{2} A^2 q''(A) \tag{2}$$

subject to the following boundary conditions $q(A_d) = 0$ and $\lim_{A \to A^*} q(A) = \dfrac{1}{1+r}$. The first boundary condition follows from the assumption that after a default bond holders are not repaid and the market closes, so the price of bonds is zero. The second boundary condition states that the price of a riskless bond is $(1 + r)^{-1}$ where A^* is the safety productivity level[6].

Value of firm's: As in a standard asset pricing model in continuous time the value of a representative firm is linked to the trend in its fundamental value. In this particular case the firm's value is determined by a regime-switching stochastic process. If there is no default

[5] This is a reduced form of an enforcement mechanism or an optimal debt contract. A theoretical characterisation of the effect of enforcement on the interest rate can be found at Krasa et al. (2008). Optimal debt contracts are designed in Hvide and Leite (2010) and Mateos-Planas and Seccia (2014).

[6] This productivity level is chosen in the same way as S&P and Fitch classify bonds as AAA or Moody's as Aaa. An obligor that has issued a prime quality bond is considered as having an extremely strong capability of meeting its financial commitments. See for example Moody's (2009) and S&P (2009).

the value of a firm is made up of the instantaneous return plus the expected change in the value of the firm. The expected change depends on the probability of default $p(A)$

$$rV_{nd}(A) = \mu_{nd} A V'_{nd}(A) + \frac{\sigma^2_{nd}}{2} A^2 V''_{nd}(A) + p(A)\left[V_d(A) - V_{nd}(A)\right] \tag{3}$$

with boundary and smooth pasting conditions $V_{nd}(A_d) = V_d(A_d)$ and $V'_{nd}(A_d) = V'_d(A_d)$. If there is a default, the value of the representative firm is:

$$rV_d(A) = \mu_d A V'_d(A) + \frac{\sigma^2_d}{2} A^2 V''_d(A)$$

with boundary conditions $V_d(0) = 0$ and $V'_d(0) = 0$. Note that in case of default there are no further changes in regime or, therefore, in the value function of the firm.

To be able to compute the value of the firm when there is no default we need to solve the equation for the value of the firm in case of default and determine the probability of default. If the government defaults, the value of a firm can be solved in closed form as $V_d(A) = A^{\beta_d}$, where β_d is the positive root of $\frac{\sigma_d^2}{2}\beta^2 + (\mu_d - \frac{\sigma_d^2}{2})\beta - r = 0$, its characteristic equation.

The probability of default can be obtained by solving the following partial differential equation

$$0 = \frac{\sigma^2_{nd}}{2} A^2 p''(A)$$

which turns out to be a Gaussian distribution, with boundary conditions $p(A_d) = 1$ and $\lim_{A\to\infty} p(A) = 0$. The first boundary condition reveals that if $A \leq A_d$ then the probability of default is zero, similarly if $A \to \infty$ the probability of default is zero.

2.1 Equilibrium

Definition: the stationary equilibrium for this economy comprises a government value function $\{W_d, W_{nd}(A)\}$, a threshold rule for default A_d, and a bond price $q(A)$ such that:

i) Given bond prices, $q(A)$, the default threshold rule A_d and value functions, $\{W_d, W_{nd}(A)\}$, solve the government problem (equation 1); .

ii) Government policy satisfies, $q(A_d) = 0$ (equation 2); .

Conditions (i) and (ii) are standard. The representative firm generates beliefs as to the probability of default by observing $q(A)$to derive its value. In equilibrium, the firm's beliefs as to the probability of default coincide with the probability of default induced by government decisions.

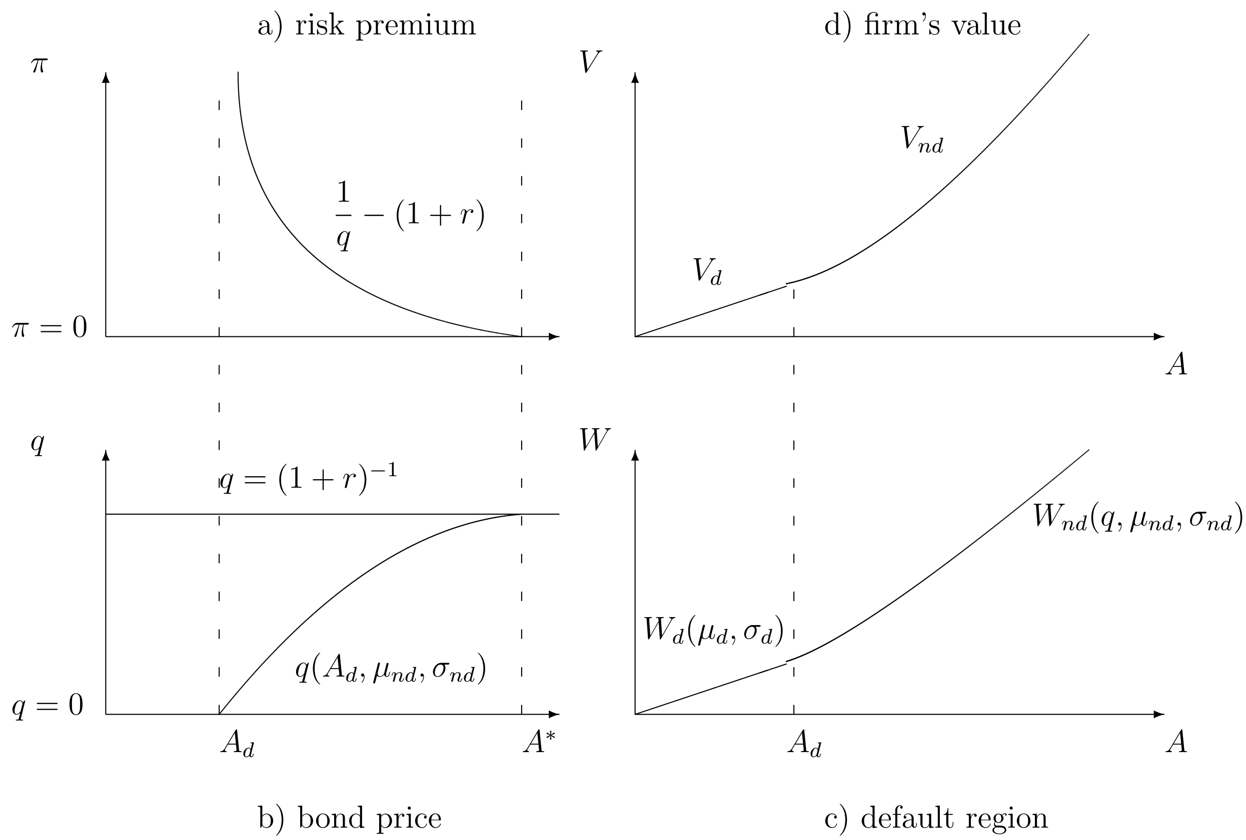


Figure 1: Risk premium, $\pi(A)$, Bonds price, $q(A)$, Default region, A_d and firms value $V(A)$. A higher risk premium implies a lower firms value

Solution: solving the stationary equilibrium entails finding the solutions to three second-order differential equations (equations 1-3). Equation 1 is a non homogeneous second order differential equation with constant coefficients, Equation 2 is a homogeneous second order differential equation with constant coefficients and Equation 3 is a non homogeneous second order differential equation with non constant coefficients. Using Laplace transforms and power series expansions equilibrium can be obtained by solving a system of linear equations (see the Appendix).

With this model it is possible to estimate the drifts and variances of the regime-switching productivity process from observables in the data. Specifically, we estimate the model using information on stock prices and risk premiums. Figure 1 illustrates the intuition. Panels (a) to (d) show key elements of the equilibrium with two key threshold values selected in their axes: the threshold value of a default, A^* and the threshold value of a risk-free bond A_d.

In panel (a) productivity is plotted against the risk premium. As productivity approaches to A_d, the risk premium diverges to infinity as the probability of repaying is zero, which causes the price of the bond to collapse to zero, as shown in panel (b). Panel (c) plots productivity against the value of the firm, which is monotonically increasing. As the risk premium is declining in productivity and the value of the firm is increasing, the model produces a negative correlation between stock prices and risk premiums. The estimation strategy is described in the next section.

3 Data and Calibration

We use data[7] on stock prices and 10Y bond yields for 9 European countries: Austria, Belgium, Finland, France, Germany, Ireland, Italy, Netherlands, Portugal and Spain.[8] These countries provide an adequate sample for estimating the parameters of our model. They belong to a free trade area with a common currency, they have similar levels of development

[7]We obtained the data from Bloomberg.

[8]We are using Germany as the risk-free option, but we do not use this country explicitly in our estimations.

and institutions and their business cycles are synchronised. Most importantly, their financial markets behaved similarly on the cusp of the European sovereign debt crisis.

Table 1: Financial series

Country	Stock market indices	10Y government bonds
Austria	ATX	GTATS10Y
Belgium	BEL20	GTBEF10Y
Spain	IBEX	GTESP10Y
Finland	HEX	GTFIM10Y
France	CAC	GTFRF10Y
Netherlands	AEX	GTNLG10Y
Ireland	ISEQ	GTIEP10Y
Italy	FTSEMIB	GTITL10Y
Portugal	PSI20	GTPTE10Y
Germany		GTDEM10Y

Table 1 shows the label of the Bloomberg series that we use. We use daily data from 2009 to 2012. We pick the most important stock index for each country and 10Y government bonds. Stock indices are normalised so that 3/1/2008=100. We compute series for the probability of default in each country, $P_j = 1 - \frac{R_j}{R_{ger}}$, using Germany as the risk-free option, where R stands for the interest rate of the 10-Y Bond in each country, therefore Germany is considered to to have a probability of default of zero.

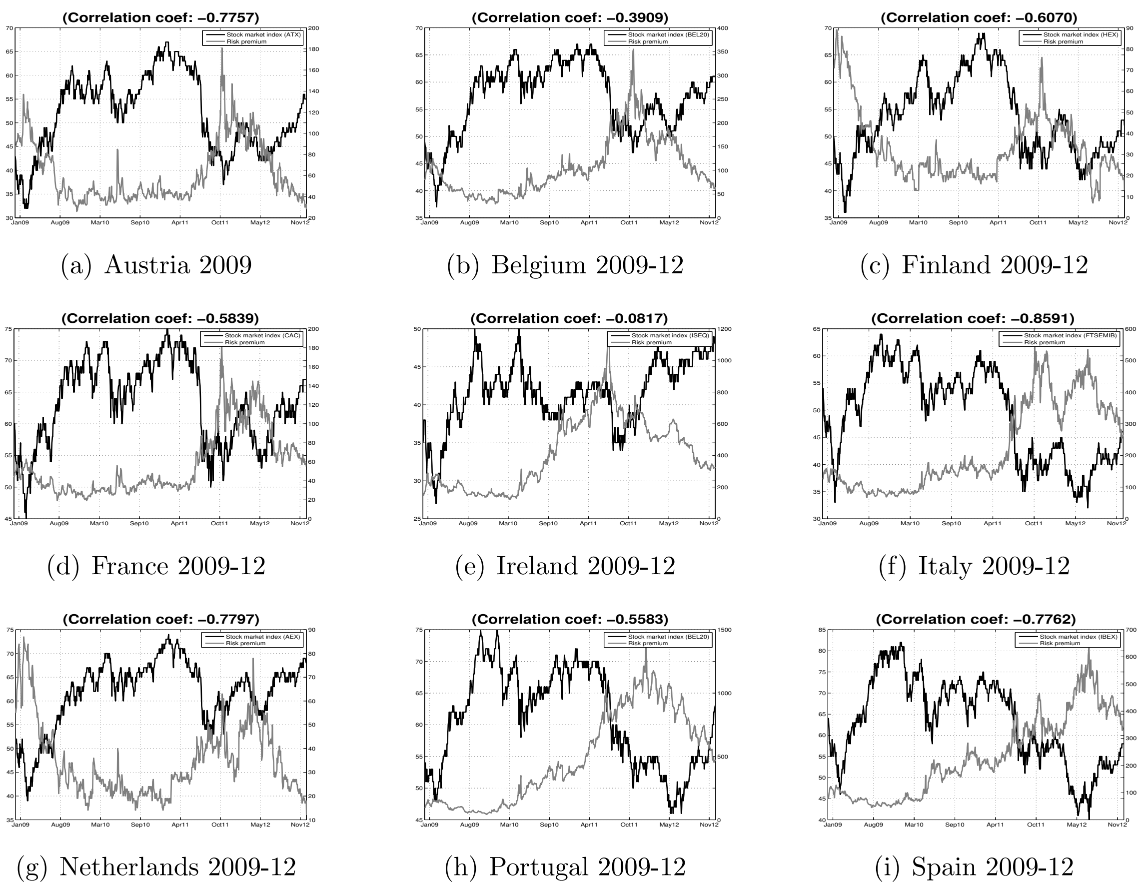
We also need fiscal data to feed into our model: taxes, debt and government expenditure as a proportion of GDP. We find the information that we need on the IMF's World Economic Outlook Database. The risk free interest rate is set to 2.86%. Table 5 displays the statistics computed. The data show that there is substantial heterogeneity in taxes, government expenditure and debt as a proportion of GDP, which will be exploited by the model.

Figure 2 shows the time series for stock prices and risk premiums for the countries in our sample and presents their correlation. An interesting observation that is exploited in the calibration is a substantial negative correlation between stock prices and risk premiums between 2009 and 2012 for the countries in the sample. This correlation turns to be close to

Table 2: Fiscal Policy Parameters

	Aut	Bel	Fin	Fra	Ire	Ita	Ndl	Por	Spa
T/GDP	48.50	49.49	53.83	49.21	33.88	46.15	45.20	40.73	36.33
G/GDP	52.63	53.77	54.77	56.77	47.71	49.81	50.80	49.76	46.65
$b = B/GDP$	69.19	97.78	49.00	79.19	117.12	120.80	60.76	122.99	84.08

Figure 2: Stock Prices and Risk Premium: Stock Prices and Risk Premium: Stock Price Index (black line) and Risk premium, obtained from 10-Y Bond interest rates (grey line). Period sample January, 1st 2009, December, 31st 2012



(a) Austria 2009 (b) Belgium 2009-12 (c) Finland 2009-12

(d) France 2009-12 (e) Ireland 2009-12 (f) Italy 2009-12

(g) Netherlands 2009-12 (h) Portugal 2009-12 (i) Spain 2009-12

-1 for almost every country for some sub-period of time. Therefore in times when the risk of default is high the value of stocks drops. This feature is helpful in calibration.

Calibration: there are two sets of parameters that are key in solving the model. The first comprises the stock of bonds issued, b, the risk free interest rate,r and taxes as a proportion of GDP, $\frac{T}{GDP}$. These are exogenous parameters which are directly imposed, and which the government takes as given in making its default decision. We solve the model many times to calibrate the parameters that characterise the regime-switching stochastic process: μ_d, μ_{nd}, σ_d and σ_{nd}. We also choose a tax rate, τ, that is consistent with the concept of government budget balance that we use to define government value functions.

The tax rate, τ, is identified through the smooth pasting condition in equation 1, which sets a value for the marginal revenue of the government at Ad as a function of the parameters of the stochastic process in the default region and the tax rate. Assuming that marginal revenues are equal to the average revenues of the government in the data, the tax parameter can by found by equating the marginal value of repaying at the boundary of the default region with T/GDP

$$W'(A_d) = \frac{\tau}{r - \mu_d + \sigma_d^2/2} = T/GDP.$$

Our results do not rely on the assumption that the government's marginal revenues are equal to its average revenues: better estimates of marginal tax rates, as in McDaniel (2011),), are not far enough away from average taxes to matter in the period of time that we are considering.re not so far-off from average taxes to matter in the period of time we are considering.

The parameters of the stochastic productivity process μ_{nd}, σ_{nd}, μ_d, σ_d are chosen to minimise the square deviation of normalised[9] stock prices and the variance of stock prices. We solve this problem by simulating the model repeatedly and following the steps of this algorithm:

Calibration Algorithm: Given b, r and an initial guess at the parameters of the stochastic productivity process $\{\mu_{nd}, \sigma_{nd}, \mu_d, \sigma_d\}$:

1. We compute $\tau = [r - \mu_d + \sigma_d^2/2]\,(T/GDP)$.

[9] We normalise both simulated series and data at the beginning of our estimation period for each country.

2. Given τ, we compute the default threshold A_d by solving equation (1).

3. Given A_d we use equation (2) to compute the $x_t = \log\left(\frac{A_t}{A_d}\right)$ that match the risk premium series data.

4. Given x_t we simulate productivity series $A_t = A_d e^{x_t}$.

5. We simulate V_t by solving equation (3).

6. We compute the objective function of the minimization routine: a) the mean quadratic deviation of the simulated stock prices series from data; b) the quadratic deviation of the standard deviation of the simulated stock prices series from data; c) the quadratic deviation of the drift of the simulated productivity series from the initial guess; and d) the quadratic deviation of the volatility of the simulate productivity series from the initial guess.

7. We use a minimisation routine to update the parameters of the stochastic productivity process.

To implement the details of the algorithm we rely on two series of data and a reduction in the dimensionality of the parameter space. We use the risk premium $\hat{\pi}_t$ and stock prices $\hat{V}_t$ as described in the previous section. In order to simplify the calibration, we guess a value for the drift and variance of the stochastic productivity process in case of no default: μ_{nd} and σ_{nd}. This is a harmless simplification for our purpose, given that we are interested in how much drift and volatility would change in case of a default. Therefore our minimisation routine searches for μ_d and σ_d.

Step 1 of the algorithm is trivial as we have the equation that computes τ, given values for μ_d and σ_d. In step 2 we compute the value of the default region in equilibrium, A_d, by solving equation 1. We need series for productivity that are consistent with observations in the data. In step 3 we use A_d, $\hat{\pi}_t$ and the solution of the equation that determines the probability of default, $p(A)$, to work out a series for productivity consistent with the observed risk premiums, $\hat{A}_t$, which can be written as:

$$\hat{A}_t = p^{-1}\left(\frac{1}{\hat{\pi}_t + (1+r)} A_d, \mu_{nd}, \sigma_{nd}\right)$$

Note that we do not use information for μ_d and σ_d to compute $\hat{A}_t$ as the price of a bond is zero in the default region, so it is not affected by the nature of the stochastic process when it switches regimes. We write down the drift and the volatility of this process as $\hat{\mu}_{nd}$ and $\hat{\sigma}_{nd}$.

In step 5, given $\hat{A}_t$ and our guesses for $\hat{\mu}_{nd}$ and $\hat{\sigma}_{nd}$, we solve equation 3 to obtain a value of the firm, V_t, consistent with the evolution of its fundamental value. Note that this value is also consistent with the information contained in the observed risk premium. Finally we construct an objective function for our minimisation routine as described in step 6. This objective function consists of the mean quadratic deviation of the simulated value of the firm relative to the stock prices in the data

$$\frac{1}{T}\sum_{t=1}^{T}\left(V_t - \hat{V}_t\right)^2$$

and we augment the objective function with three additional moments: the quadratic deviation of the volatility of the simulated firm value relative to the volatility of stock prices, the quadratic deviation of the drift of the productivity process,$\hat{\mu}_{nd}$, relative to our guess, μ_{nd}, and the quadratic deviation of the volatility of the productivity process, $\hat{\sigma}_{nd}$ relative to our guess σ_{nd}. The algorithm stops when μ_d and σ_d such that the objective function is minimised to a certain degree of precision. With the parameters estimated we can measure how much TFP falls in case of a default.

4 Results

This section presents the results of our calibration and evaluates how "good" the fit of the model is. To that end we apply our model to the study of four important issues. The first criterion for goodness of fit is whether the model produces sensible predictions as to how much GDP falls after a default. The second criterin is a comparison of whether the model produces rates of growth of GDP after a default that are compatible with the rates of growth in the countries in our sample after the sovereign debt crisis. As countries grow after a crisis, recovery is inevitable. Recovering the previous levels of GDP is a question of time. Therefore, the third criterion is whether the model is capable of producing a reasonable distribution of recovery times. To test this criterion we examine Argentina's recovery from its default in 2002. A theoretical prediction of general equilibrium models of default is that more indebted countries have larger default regions. We use our model and cross country variations of debt to GDP to study whether this fits the theory. The fourth and final criterion is to examine whether our model is consistent with the theoretical prediction that default zones shrink with reductions in the level of debt of a typical country.

Baseline productivity process. Before we can evaluate whether our model produces credible drops in productivity we need to look at whether the calibrated stochastic productivity process is reasonable for the group of countries that we study when there is no default. Our main target is to calibrate the productivity process of a typical country, so we pool all countries (and all years) and run the calibration algorithm[10].

The last column of Table 3 shows that a typical country's productivity rate of growth is 2.39% in nominal terms, a number that seems reasonable given that the average inflation rate of the countries in our sample is 1.57%. The productivity growth rate across countries seems to be fairly constant and does not seem to be related to idiosyncratic volatilities.[11] Volatilities are

[10]We use the average debt to GDP and taxes to GDP ratios for the countries in our sample as exogenous inputs for our calibration algorithm.

[11]Danthine and Jin (2007) show that financial volatility is a multiple of macroeconomic volatility.

Table 3: Baseline Calibration

	Aut	Bel	Fin	Fra	Ire	Ita	Ndl	Por	Spa	**Pooled**
μ (%)	2.3907	2.3899	2.3873	2.3897	2.3899	2.3870	2.3879	2.3898	2.3850	**2.3923**
σ (%)	3.3756	5.3386	2.0847	4.8034	6.1940	5.0912	2.0436	7.0856	6.0494	**4.3436**

quite heterogeneous across countries but this comes as no surprise, as countries face different yield curves and issue and restructure their debt with different maturity structures. Cunha (2013) highlights that countries with shorter debt maturities face a higher risk of rollover that may be captured in idiosyncratic volatilities. Arellano and Ramanarayanan (2012) also record a negative correlation between the maturity of debt and bond spreads. Eurostat reports that in 2014 roughly 40% of Spain's debt had maturity periods of less than 7 years, whereas 92% of Finland's debt matured at more than 15 years.

The within sample fit of the model is reasonable. We evaluate the distribution of errors, defined as the difference between the model prediction and the stock prices in the data. To summarise key statistics we rely on the use of box-plots for the distribution of these errors. Figure 4 presents box-plots for each country, assuming them to be endowed with the pooled calibration stochastic process. Most of the errors fall within the 10% bands from zero, the median is very close to zero for most countries and there are not many outliers in general. Measuring the cost of a default by comparing how much the drift would have fallen does not therefore seem a far-fetched experiment.

Defaults and productivity drops. Our measure of the instantaneous cost of a default, in terms of productivity, is the ratio of the drift in case of default relative to no-default: $\frac{\mu_d}{\mu_{nd}}$. Table 4 shows that productivity falls by 3.70% for a typical country, although there is some cross-country heterogeneity which reflects, among other things, differences in indebtedness and tax proceeds over GDP across the countries in our sample.

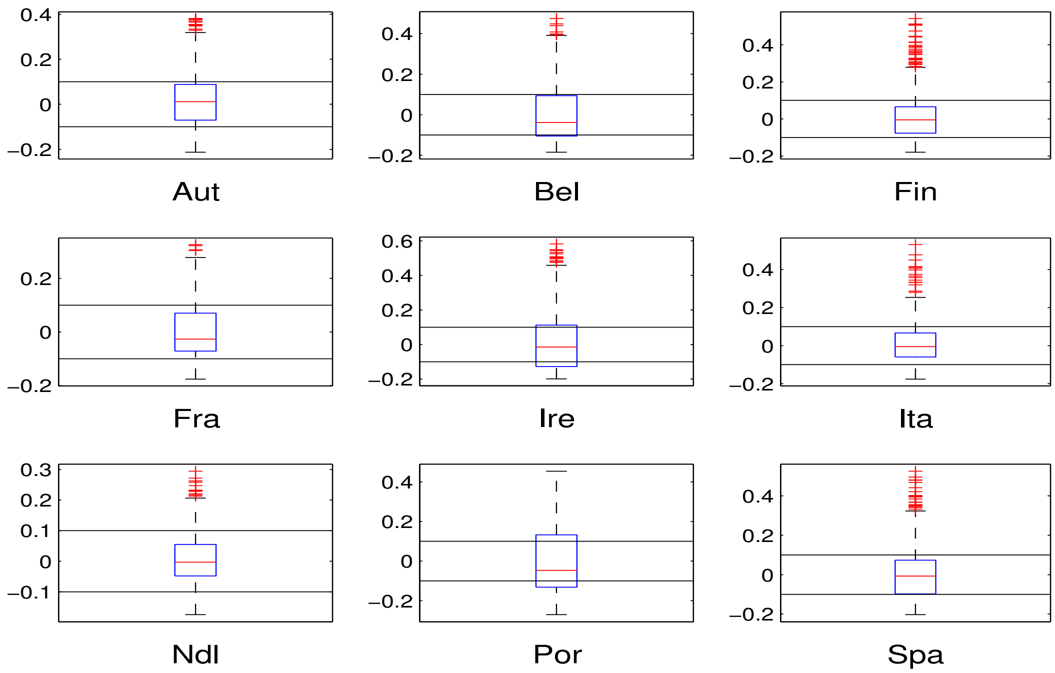


Figure 3: ror distribution: 10% bands. In each box, the central mark is the median; the edges of the box are the 25th and 75th percentiles. The red cross represents outliers. The whiskers extend to the most extreme data points not considered as outliers.

model provides an estimate of the productivity cost of a default consistent with many general equilibrium models in the literature, which typically adopt a figure of 5%. The paper by Cole and Kehoe (2000) is one of the first to target a 5% drop in productivity for a 2% probability of default in the case of Mexico. Da-Rocha et al. (2013) assumes the same figure for Argentina. Nuño Barrau and Thomas (2015) target an output loss of 6% for the European Monetary Union and Arellano and Ramanarayanan (2015) target 4.5% in a study for Brazil. By comparison, our model predicts a productivity fall of 3.70%. We therefore find a very similar default cost using a different model to exploit a different source of information, with different countries and in a different period.

Our findings can be compared with papers that study the cost of default in private business, particularly in the US. Davydenko et al. (2012) estimate that the cost of default is 21.70% of a

Table 4: Drops

	Aut	Bel	Fin	Fra	Ire	Ita	Ndl	Por	Spa	Pooled
$\Delta\mu$	97.03	99.63	68.29	99.98	98.73	97.35	86.42	99.99	99.66	96.30
$\Delta\sigma$	45.21	21.39	75.42	23.84	18.56	23.26	97.02	16.16	18.85	26.15

$\Delta\mu = \mu_d/\mu_{nd}$
$\Delta\sigma = \sigma_d/\sigma_{nd}$

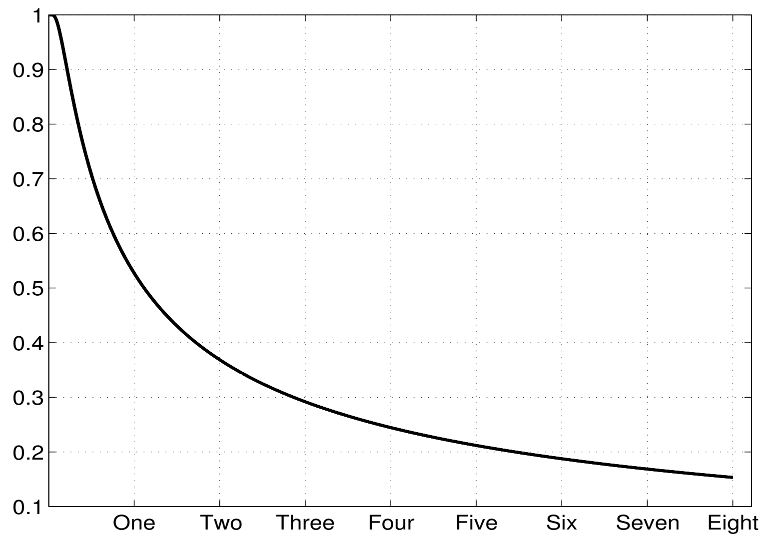
Firm's value. This figure can be compared with our estimates if it is assumed that production is characterised by a labour-augmenting Cobb-Douglas technology $y = A^{1-\alpha}k^{\alpha}l^{1-\alpha}$, where the price of a firm is the value of its capital stock, according to the neoclassical growth model. In equilibrium, the capital stock is proportional to productivity: $k \approx A^{\frac{1}{1-\alpha}}$. Therefore, for a typical value of $\alpha = .36$, we find that a drop in the value of the firm of a 21.70% is equivalent to a fall of 14.40% in productivity.[12]

Defaults and recovery time. Defaults in sovereign debt are typically associated with output drops and sudden stops, where GDP falls at the time of default and growth subsequently resumes at a slower rate. This enables us to test the goodness of our model in different dimensions, and run a second test of the model estimates by testing the implications for the behaviour of GDP.

Ito's Lemma can be invoked to derive a stochastic process to describe GDP, assuming it is characterised by a Cobb-Douglass labour augmenting technology. The drift of this Brownian motion, μ_y, is equal to $\mu_y = (1-\alpha)\left(\mu_A - \frac{\alpha}{2}\sigma_A^2\right)$ and the standard deviation σ_y is equal to $\sigma_y = (1-\alpha)\sigma_A$. For a value of $\alpha = .36$, it ispossible to derive how far GDP falls instantly at default as $\frac{\mu_y^d}{\mu_y^{nd}}$; μ_y^d will be the rate at which a country grows after default. Our model predicts that GDP will fall by 3.71% for a typical country. Between 2008-2009 GDP fell by 4% on average, so the predicted fall in GDP in our model is consistent with data averages. The

[12] Glober (2013) finds default costs by industry in the range of 0.35-0.53, equivalent to a fall in TFP in the range of 0.20-0.35.

Figure 4: Time to Recover



third test of our model consists of comparing the expected growth after default in our model, 1.17%, with that which actually took place in the countries in the sample. In 2014 countries resumed growth at a rate of 1.4%. Our model therefore provides a reasonable description of the crisis and recovery experience of a typical country in terms of GDP.

As growth resumes after a crisis, recovery is inevitable. The fourth measure of the goodness of the model is whether it provides a reasonable distribution of recovery dates. We define the recovery date as the time when GDP reverts to the level prior to the default. With our model we can compute the probability distribution of a recovery in closed form.[13] To illustrate this idea we plot, in Figure 4, plots the distribution of GDP recovery dates when Argentina is

[13]Let $\bar{x} = \log(\frac{y}{y_d})$ (where y_d is the default threshold in terms of GDP) be a random variable. In this case, recovery is defined as $\bar{x} = 0$. A recovery date can be defined as $T(x) = \{T : \bar{x} \geq 0\}$ and, following Harrison (1985),the distribution of recovery dates can be written as

$$P(T(x) > t) = \left[\phi \left(\frac{x - \mu_y t}{\sigma_y \sqrt{t}} \right) - e^{\dfrac{2\mu_y x}{\sigma_y^2}} \phi \left(\frac{-x - \mu_y t}{\sigma_y \sqrt{t}} \right) \right]$$

where ϕ is a $N(0,1)$ distribution function.

examined via our pooled productivity process. In 2001, Argentina's GDP fell by 20%. The probability of recovery after two years in the default region was 2/3, so in our model there is a fair chance of a fast recovery but a long-lasting recovery such as that experienced by the Greek economy is not ruled out.Guido and Werning (2013) build a model where there are slow moving crises to account for the European sovereign bond crises, compared to rollover crises like that of Argentina. Our model is able to accommodate both as there is still a 20% chance of not recovering seven years later.

Default zones and initial debt. The fifth and last example of the goodness of the model exploits cross-country differences in the debt to GDP ratio and differences in taxes as a proportion of GDP. A theoretical prediction of models of sovereign default such as that of Cole and Kehoe (2000), is that the size of the default zone increases monotonically with the level of debt. A nice feature of our calibration is that in our model the default zones of different countries are positively correlated with the initial levels of debt in the data.

To quantify the relationship between debt and default regions, we endow each country with the pooled stochastic process and compute a default region, A_d, by solving the equilibrium for each country. Table 5shows that there are substantial cross-country differences in debt and taxes as a proportion of GDP, so substantial differences in the default zones should be found. This is confirmed in the last row of Table 5, , which shows the equilibrium default

Table 5: Default Zones

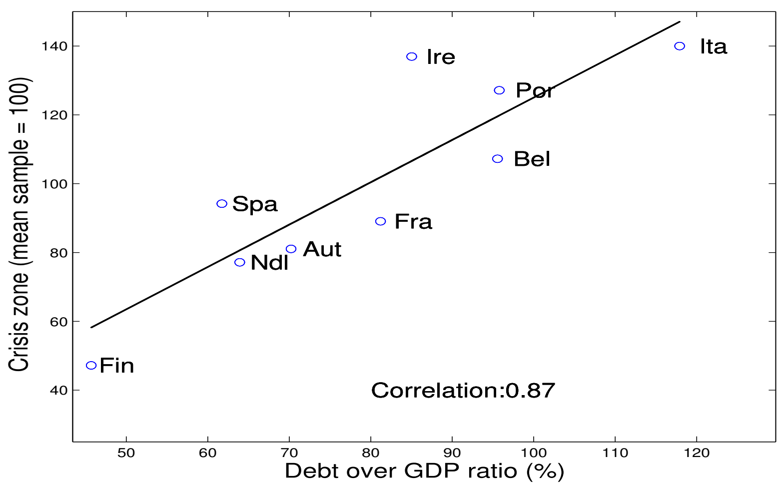
	Aut	Bel	Fin	Fra	Ire	Ita	Ndl	Por	Spa
A_d	81.06	107.25	47.20	89.08	136.93	140.00	77.15	127.14	94.20

zones for our sample of countries. The magnitude of these raw numbers is hard to grasp, so we normalise the average of the default regions to 100 and plot it against debt-to-GDP in Figure 5. This confirms the positive link between debt and the size of the default zone.

Finland has a default zone 40% smaller than the average with a debt-to-GDP ratio of 49%, whereas Italy has a default zone 40% larger than the average for a ratio of 120%.

Default zones and debt consolidation. Debt consolidation was standard policy advice during the European sovereign debt crisis. However, the quantitative impact of debt consolidation on the probability of default and its cost is subject to much controversy. The positive correlation between debt and default zones can be exploited to shed some light on the issue. Figure 5 depicts a regression which shows the expected default zone, $A_{d,i}$, for a given level of debt, b_i

Figure 5: Debt and Default Zones

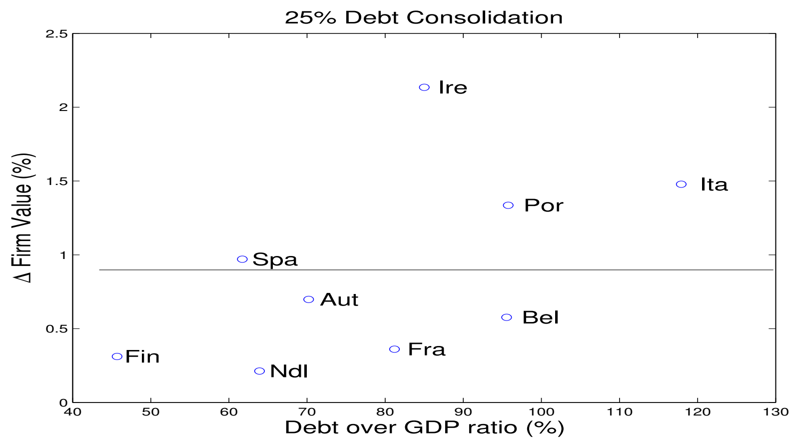


$$A_d = \beta_0 + \beta_1 b$$

A debt consolidation policy, ∇b, can be imposed to measure how far the default zone is expected to drop, , $\hat{A}_d = A_d + \beta_1 \nabla b$. With this new threshold we compute a probability of default $\hat{p}_d = p_d \left(1 - \frac{\beta_1 \nabla b}{A_d}\right)$ which implies a risk premium of $\hat{\pi}_d = \frac{\hat{p}_d}{p_d}\left(\frac{1-p_d}{1-\hat{p}_d}\right)$. We also compute a new equilibrium value of stocks, keeping the same stochastic process. It is possible to find a new log productivity of a firm and $V(\hat{x}) - V(x)$, the change in the value of firms induced by debt consolidation.

We find that a reduction of 10% in debt increases the value of firms by 0.35%, or 0.02% of GDP on average. Of course this effect is heterogeneous across countries. An example of this

Figure 6: Debt Consolidation



heterogeneity is displayed in Figure 6, which shows what the impact of a debt consolidation of 25% would have been on the value of firms, given the initial stock of debt[14]. Consider Spain, an otherwise average country. If Spain had reduced its debt by 25% if would have increased the value of its firms by 1%, which would have translated into an increase of .65% in GDP. Flipping the argument around, we conclude that attempts to mitigate the effect of the sovereign debt crisis through fiscal expansions had little effect on the economy. Our model implies small fiscal multipliers, yet another observation that seems consistent with the data.

5 Conclusions

In this paper we extract information from financial data (stock prices and risk premiums) about how big the productivity costs of a default shock may be. We use a sample of European countries affected by the region's sovereign debt crises. Our method consists of estimating

[14]The solid line represents the average

regime-switching productivity parameters, drifts and variances, using a continuous time model of government default decisions under uncertainty.

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A Appendix

This appendix presents the solution of the second order differential equations 1, 2 and 3.

Solution of Equation 2. For any $A_d > 0$, and using $x = \log\left(\frac{A}{A_d}\right)$, the debt price is the solution of the boundary-value problem that consists of solving the equation:

$$-rq(x) + \hat{\mu}_{nd}q'(x) + \frac{\sigma_{nd}^2}{2}q''(x) = 0 \tag{4}$$

with boundary conditions $q(0) = 0$ and $q'(0) = k$, where $\hat{\mu}_{nd} = \mu_{nd} - \frac{1}{2}\sigma_{nd}^2 < 0$ and k is an arbitrary constant. We solve the boundary-value problem using Laplace transforms, $\mathscr{L}[q(x)]$. Laplace transforms are given by

$$\begin{aligned}\mathscr{L}[q'(x)] &= s\mathscr{L}[q(x)] - q(0),\\ \mathscr{L}[q''(x)] &= s^2\mathscr{L}[q(x)] - sq(0) - q'(0).\end{aligned}$$

By applying Laplace transforms in equation (4)

$$(\frac{\sigma_{nd}^2}{2}s^2 + \hat{\mu}_{nd}s - r)\mathscr{L}[q(x)] - (s + \hat{\mu}_{nd})q(0) - \frac{\sigma_{nd}^2}{2}q'(0) = 0 \tag{5}$$

and the boundary condition $g(0) = 0$, we obtain:

$$\mathscr{L}[q(x)] = \frac{\sigma_{nd}^2}{2}\frac{k}{(s - z_1)(s - z_2)},$$

where $z_i = \left(-\hat{\mu}_{nd} \pm \sqrt{\hat{\mu}_{nd}^2 + 2r\sigma_{nd}^2}\right)\left(\sigma_{nd}^2\right)^{-1}$, $i = 1, 2$. We obtain the solution by solving the laplace inverses given by:

$$q(x) = \mathscr{L}^{-1}\left[\frac{k\sigma_{nd}^2/2}{(s - z_1)(s - z_2)}\right] = \frac{k\sigma_{nd}^2/2}{(z_1 - z_2)}\left(e^{z_1x} - e^{z_2x}\right) = \frac{1}{1 + r}\left(\frac{e^{z_2x} - e^{z_1x}}{e^{z_2\overline{x}} - e^{z_1\overline{x}}}\right) \tag{6}$$

and taking into account the second boundary condition, $\lim_{x\to\overline{x}} q(x) = \frac{1}{1+r}$ where $\overline{x} = \log\left(\frac{A^*}{A_d}\right)$.

Solution of Equation 1. Equation default regions are characterised by the non-homogeneous second-order differential equation

$$rW(x) - \hat{\mu}_{nd}W'(x) - \frac{\sigma_{nd}^2}{2}W''(x) = \tau A_d e^x + [q(x) - 1]b$$

with boundary conditions $W(0) = \dfrac{\tau A_d}{r - \mu_d + \sigma_d^2/2}$ and $W'(0) = \dfrac{\tau A_d}{r - \mu_d + \sigma_d^2/2}$. Taking the Laplace transform of both sides of the differential equation defaults regions are characterising by solving

$$\left(r - \hat{\mu}_{nd}s - \frac{\sigma_{nd}^2}{2}s^2\right)\mathscr{L}[W(x)] = -\left(\hat{\mu}_{nd} + \frac{\sigma_{nd}^2}{2}s\right)W(0) - \frac{\sigma_{nd}^2}{2}W'(0) - \frac{b}{s} + \frac{\tau A_d}{s - 1} + b\mathscr{L}[q(x)],$$

where

$$\mathscr{L}[q(x)] = \frac{1}{(1+r)(e^{z_1\overline{x}} - e^{z_2\overline{x}})}\left[\frac{1}{s-z_1} + \frac{1}{s-z_2}\right].$$

$H(s) = \mathscr{L}[W(x)]$ satisfies,

$$H(s) = \frac{P_1 + P_2 s + P_3 s^2 + P_4 s^3 + P_5 s^4 + P_6 s^5}{s(s-1)(s-z_1)^2(s-z_2)^2}$$

where the vector $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 1+z_1+z_2 & 0 & 0 & 0 \\ 1+z_1+z_2 & -z_1z_2+z_1+z_2 & -1 & 1 & 0 \\ -(z_1z_2+z_1+z_2) & z_1z_2 & 1+z_1+z_2 & -(z_1+z_2) & -(z_2-z_1) \\ z_1z_2 & 0 & -(z_1z_2+z_1+z_2) & z_1z_2 & z_2-z_1 \\ 0 & 0 & z_1z_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mu}_{nd}W(0) + \frac{\sigma_1^2 W'(0)}{2} \\ \frac{\sigma_1^2 W'(0)}{2} \\ b \\ \tau A_d \\ \frac{(1+r)b}{\left(e^{z_2\overline{x}} - e^{z_1\overline{x}}\right)} \end{bmatrix}$$

Expanding $H(s)$ in partial fractions

$$H(s) = \frac{C_1}{s} + \frac{C_2}{s-1} + \frac{C_3}{(s-z_1)} + \frac{C_4}{(s-z_2)} + \frac{C_5}{(s-z_1)^2} + \frac{C_6}{(s-z_2)^2},$$

Applying the laplace inverses given by:

$$W(x) = \mathscr{L}^{-1}[H(s)] = C_1 - C_2 * e^x + C_3 x e^{z_1 x} + C_4 x e^{z_2 x} + C_5 x^2 e^{2z_1 x} + C_6 x^2 e^{2z_2 x}$$

we can find the the solution of $W(x)$ by solving a system of linear equations which can be written in matrix notation as:

$$\begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{bmatrix}^T = \mathbf{\Lambda}^{-1}\mathbf{P}$$

and $\mathbf{\Lambda}$ is equal to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -(2z_1+z_2+1) & -2(z_1+z_2) & -(2z_2+z_1+1) & -(2z_2+z_2+1) & 1 & 1 \\ 4z_1z_2 + z_1^2z_2^2 + 2(z_1+z_2) & 4z_1z_2 + z_1^2 + z_2^2 & z_2(z_2+2z_1) + 2z_2 + z_1 & z_1(z_1+2z_2) + 2z_1 + z_2 & -(1+2z_1) & -(1+2z_2) \\ -2(\frac{z_1^2}{2} + z_2 + z_1z_2^2 + 2z_1z_2) - z_2^2 & -2(z_1z_2^2 + z_2z_1^2) & -(z_1z_2 + z_2(z_2+2z_1)) & -(z_1z_2 + z_1(z_1+2z_2)) & z_1(1+2z_1) & z_2(1+2z_2) \\ z_1^2z_2^2 + 2(z_1z_2^2 + z_1^2z_2) & z_1^2z_2^2 & z_1z_2 & z_1z_2 & -z_1^2 & -z_2^2 \\ -z_1^2z_2^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Given μ_{nd}, σ_{nd}^2 μ_d, σ_d^2 and r, b and τ, A_d, is obtained by solving $W(x) = \mathscr{L}^{-1}[H(s)]$ at $x = 0$, i.e.

$$W(0) = W(x)|_{x=0} = C_0(A_d) - C_1(A_d) = \frac{\tau A_d}{r - \mu_d + \sigma_d^2/2}. \tag{7}$$

Solution of Equation 3. To solve firm value if the goverment has not defaulted $V_{nd}(A)$, we rewrite the switching problem throughthe following change of variable g $x = \log\left(\frac{A}{A_d}\right)$

$$\left[r - e^{-\left(1-\frac{2\mu_{nd}}{\sigma_{nd}^2}\right)x}\right] V_{nd}(x_t) = \hat{\mu}_{nd}V'_{nd}(x_t) + \frac{\sigma_{nd}^2}{2}V''_{nd}(x_t) + e^{-\left(1-\frac{2\mu_{nd}}{\sigma^2}\right)x} A_d^{\beta_d} e^{\beta_d x} \tag{8}$$

where boundary conditions are given by $V_{nd}(0) = A_d e^{\beta_d}$ and $V'_{nd}(0) = \beta_0 A_d e^{\beta_d}$, and the probability of defaulting is $e^{-\left(1-\frac{2\mu_{nd}}{\sigma^2}\right)x}$. We solve equation 8 with a power series expansion. The basic idea is similar to that in the method of undetermined coefficients: We assume that the solutions of a given differential equation have power series expansions, and then we attempt to determine the coefficients so as to satisfy the differential equation. Rewrite equation 8 as

$$\left[r - e^{(-a_0 x)}\right] V - a_1 V' - a_2 V'' = a_3 e^{bx}. \tag{9}$$

We use the notation $V = V_{nd}(0)$. Consider an Taylor expansion

$$V(x) = V + V'x + \sum_{k=2}^{n} \frac{1}{k!} V'^{(n)} x^n.$$

Differentiating equation (9) n times yields a linear system

$$\begin{bmatrix} \lambda_{1,0}(r-1) & -a_1 & -a_2 & 0 & ... & 0 & 0 & 0 \\ \lambda_{2,0}a_0 & \lambda_{2,1}(r-1) & -a_1 & -a_2 & ... & 0 & 0 & 0 \\ \lambda_{3,0}a_0^2 & \lambda_{3,1}a_0 & \lambda_{3,2}(r-1) & -a_1 & ... & 0 & 0 & 0 \\ \lambda_{4,0}a_0^3 & \lambda_{4,1}a_0^2 & \lambda_{4,2}a_0 & \lambda_{4,3}(r-1) & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ \lambda_{n+1,0}a_0^{n+1} & \lambda_{n+1,1}a_0^n & \lambda_{n+1,2}a_0^{n-1} & ... & \lambda_{n+1,n-1}a_0 & \lambda_{n+1,n}(r-1) & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} V \\ V' \\ V'' \\ V'^{(3)} \\ ... \\ V'^{(n)} \\ V'^{(n+1)} \\ V'^{(n+2)} \end{bmatrix} = a_3 \begin{bmatrix} 1 \\ b \\ b^2 \\ b^3 \\ ... \\ b^{n+1} \end{bmatrix}$$

where $\lambda_{n,j} = (-1)^{n+j+1} \begin{pmatrix} n! \\ j! \end{pmatrix}$ are the Pascal's triangle numbers (in absolute value). Given this recurrence relationship, the successive coefficients can be evaluated one by one by writing the recurrence relationship first for n = 0, then for n = 1, and so on. Therefore, the solution is merely a function of the boundary conditionsV_0 and V'_0, i.e.

$$\begin{bmatrix} V'' \\ V^{(3)} \\ V^{(4)} \\ V^{(5)} \\ ... \\ V^{(n+2)} \end{bmatrix} = \begin{bmatrix} -a_2 & 0 & ... & 0 & 0 & 0 \\ -a_1 & -a_2 & ... & 0 & 0 & 0 \\ (r-1) & -a_1 & ... & 0 & 0 & 0 \\ 3a_0 & (r-1) & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ a_0^{n-1} & ... & na_0 & (r-1) & -a_1 & -a_2 \end{bmatrix}^{-1} \left(a_3 \begin{bmatrix} 1 \\ b \\ b^2 \\ b^3 \\ ... \\ b^{n-1} \end{bmatrix} - \begin{bmatrix} (r-1) & -a_1 \\ a_0 & (r-1) \\ -a_0^2 & 2a_0 \\ a_0^3 & -3a_0^2 \\ ... & ... \\ a_0^{n+1} & (n+1)a_0^n \end{bmatrix} \begin{bmatrix} V_0 \\ V'_0 \end{bmatrix} \right).$$